

## Realistic on-axis fields for the RFOFO cooling ring

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We calculate the on-axis field components for the RFOFO cooling ring. The ring consists of 12 cells, each containing two solenoids. Each solenoid is modeled here as a current sheet. Independent calculations made using the Biot-Savart equation give identical results. The dipole field comes from tipping the solenoid axis above and below the bending plane. We show the field components in the accelerator coordinate system. We checked the validity of the solution by showing that the field components satisfy Maxwell's equations along the circle. This was done using independent calculations in cylindrical and Cartesian coordinates.

### 1. Introduction

The RFOFO cooling ring was conceived by Bob Palmer and represents a promising approach to achieving 6D cooling for a neutrino factory or muon collider. The original design [1] used a field on the reference particle trajectory (RPT) consisting of an alternating solenoid field along the trajectory and a combined function dipole field transverse to it. The dipole field was 0.125 T and the field index was 0.5 everywhere along the RPT. Later an approximate field configuration was considered [2] where the dipole field came from tipping the solenoids. Simulations for this design used periodic functions for  $B_S$  and  $B_Y$  on the RPT, but assumed the  $B_X$  field was negligible. In what follows we refer to this as the *nglr* model of the ring.

In this note we consider the exact field on the system axis (SA) for the ring. The SA is of course a circle and is not the same as the RPT. However, work by Valeri Balbekov [3] has shown that the RPT (i.e. the closed orbit for the reference particle) should be  $\sim 2$  cm from our SA, while the sheets we use here have radii  $\sim 80$  cm. Thus we expect these fields to give a good approximation to the true fields. In subsequent work we intend to find the actual fields on the RPT.

Balbekov [3] has also given on-axis fields in his note that differ from those given here. We should emphasize that our fields result from summing all 24 solenoids in the ring, whereas Balbekov's fields come from the solenoids in a given cell and its nearest

neighbor cells. Balbekov assumes that iron will be present around the solenoids that shields the axis from the other solenoids. We do not make this assumption.

## 2. Field from a current sheet

The field components ( $B_U$ ,  $B_V$ ) at some observation point O due to an annular current sheet S with radius  $a$ , length  $L$  and current density  $J$  is given by a known function [4,5]  $BS(u,v;a,L,J)$ , where  $(u,v)$  are the axial and radial distances of O in a cylindrical coordinate system centered at S. The polarity of the sheet field is determined by the sign of  $J$ . The sign convention for the coordinates is

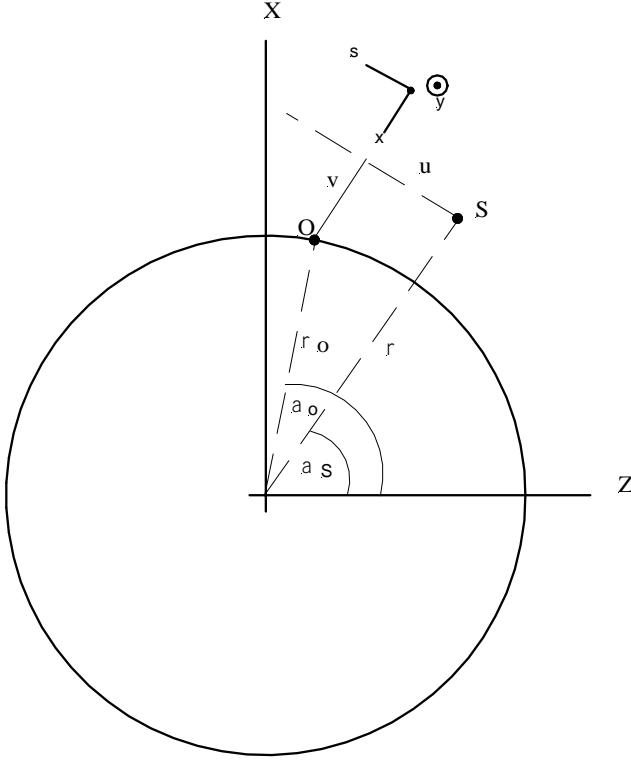
$u$  is positive if O is in the positive  $z$  direction in the frame of S  
 $v$  is always positive

For the fields

$B_U$  is positive if it points in the same direction as the polarity of the sheet  
 $B_V$  is positive if it is diverging from the axis of the sheet

## 3. Sheet fields on a ring

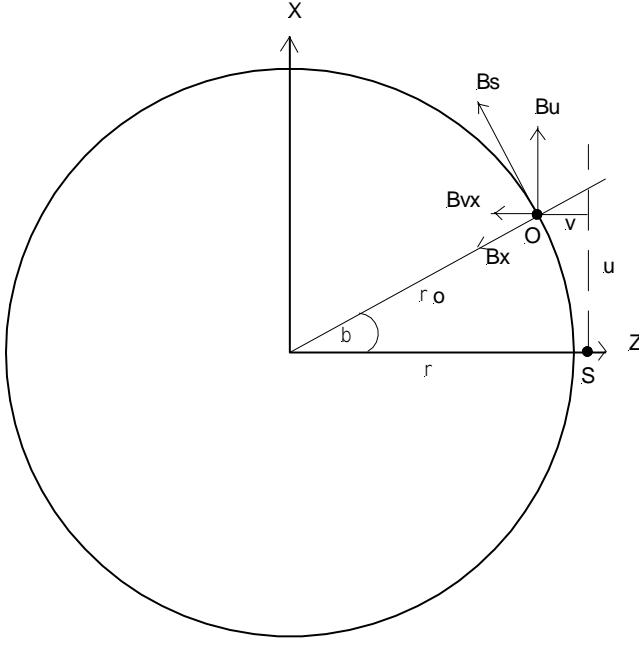
Let us consider a problem where all our observations points are constrained to lie on a circle of radius  $\rho_0$ . We define the plane containing this circle to be the *reference plane*. The  $x$  and  $z$  axes of a right-handed Cartesian coordinate system are defined to lie in the reference plane, as shown in Fig. 1.



**Figure 1.**

We assume that the centers of all current sheets lie on the reference plane. Take an observation point  $O$  located at the azimuthal angle  $\alpha_o$  and a sheet  $S$  whose center has the radius  $\rho$  and angle  $\alpha_s$ . The problem we want to solve here is to find the field at  $O$  due to the sheet at  $S$ . We want the answer in a right-handed “accelerator” coordinate system, where the  $z$  axis is tangent to the circle at  $O$ , the  $x$  axis lies along the radius at  $O$ , and the  $y$  axis is perpendicular to the reference plane.

In the case when the symmetry axis of the sheet is also in the reference plane (i.e. no dip) it is fairly easy to solve this problem directly. From symmetry the only relevant angle is the relative angle between  $O$  and  $S$ . Thus we define the angle  $\beta = \alpha_o - \alpha_s$  and for convenience locate  $S$  along the  $z$  axis, as shown in Fig. 2.



**Figure 2.**

The coordinates of O in the sheet coordinate system are

$$\begin{aligned}
 u &= \rho_o \sin \beta \\
 v_x &= -\rho + \rho_o \cos \beta \\
 v &= |v_x|
 \end{aligned} \tag{1}$$

where we keep the convention of always taking the observation point minus the sheet coordinates. The angle  $\beta$  is positive in this example. Once  $u$  and  $v$  are known, we can evaluate  $BS$  and find the field components  $B_U$  and  $B_V$ . Since  $B_V$  is the radial component in a cylindrical coordinate system, its value is independent of azimuth. We get the signed component in the plane from

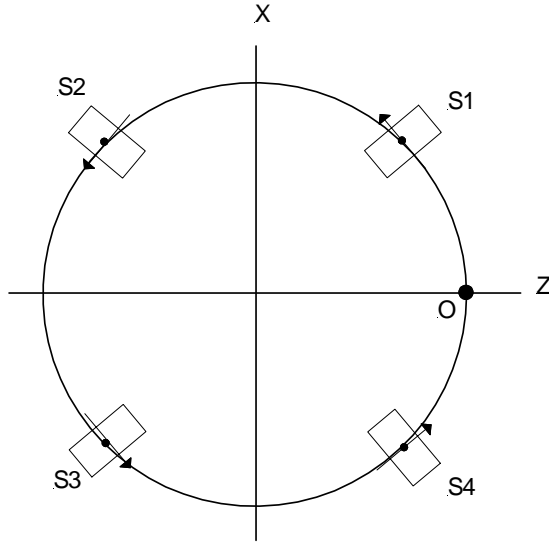
$$B_{vx} = \frac{-v_x}{v} B_V \tag{2}$$

Note that  $v_x$  is negative in this example and thus  $B_{vx}$  is positive. In order to get the proper numerical values the field components in the accelerator coordinate system are

$$\begin{aligned}
 B_x &= -B_U \sin \beta + B_{vx} \cos \beta \\
 B_y &= 0 \\
 B_s &= B_U \cos \beta + B_{vx} \sin \beta
 \end{aligned} \tag{3}$$

So the first constraint we have to satisfy is that the general transformation equations must reduce to Eqs. 1 and 3 when the dip angle out of the reference plane is zero.

Next consider the simple test example shown in Fig. 3.



**Figure 3.**

We have four sheets spaced equally around the circle and an observation point O at the right edge of the circle. The current is defined as positive for all four sheets. Table 1 gives the contributions of each of the sheets at O calculated using Eqs. 1-3.

*Table 1: Contributions of sheets to field at O.*

sheet	u	$v_x$	$b_U$	$b_V$	$\sin \beta$	$\cos \beta$	$b_x$	$b_s$
1	-3.7	-1.5	26	-17	-0.71	0.71	6.5	31
2	-3.7	-9.0	-0.7	-1.3	-0.71	-0.71	0.4	1.4
3	3.7	-9.0	-0.7	1.3	0.71	-0.71	-0.4	1.4
4	3.7	-1.5	26	17	0.71	0.71	-6.5	31

The  $b_i$  quantities are relative field components. The signs are such that all the contributions  $b_s$  add, while the  $b_x$  cancel. Thus a second constraint on the general transformation equations is that they produce field components with the proper symmetry to produce these resultant field components at O.

In the more general case where the symmetry axis of the sheet makes a dip angle  $\theta$  with respect to the reference plane, the geometry is considerably more complicated and we proceed instead by using a series of coordinate transformations. First, starting with the known coordinates of the observation point O and the known coordinates and orientation of the sheet S in the LAB coordinate system, we must find the cylindrical coordinates (u,v) of the point in the sheet's coordinate system. This will enable us to evaluate the function  $BS$  and find the field components  $B_U$  and  $B_V$  at O. We then need a second set of transformations that begin with the known field components in the sheet coordinate system and give us the desired field components at O in the LAB coordinate system.

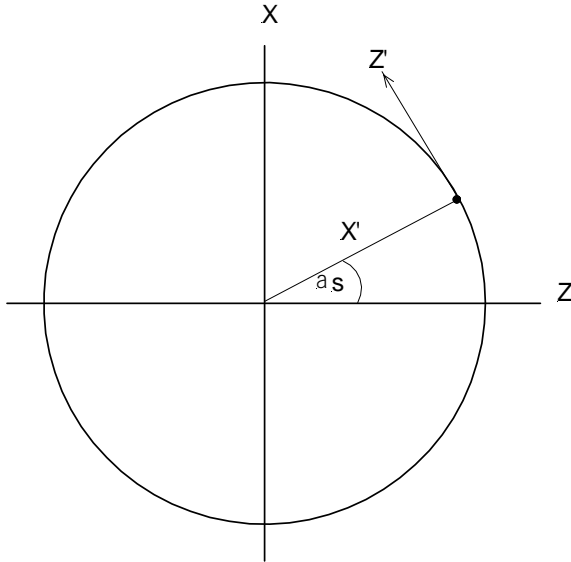
Let us start with a general observation point O. In a fixed Cartesian coordinate system O will have the coordinates

$$O = \begin{pmatrix} x_o \\ y_o \\ z_o \end{pmatrix}$$

We also start for the moment with a general sheet S. In a coordinate system translated to the center of the sheet, O has the coordinates

$$O' = \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} x_o - x_s \\ y_o - y_s \\ z_o - z_s \end{pmatrix}$$

Next let us find O in a coordinate system where the symmetry axis of the sheet is aligned with the fixed Cartesian Z axis, as shown in Fig. 4.



**Figure 4.**

We must perform the rotation  $R(-\alpha_s - \pi/2, y)$  on  $O'$ , where  $\alpha_s$  is the angular position of S in the fixed coordinate system. We find

$$O'' = \begin{pmatrix} -dx \sin \alpha_s - dz \cos \alpha_s \\ dy \\ dx \cos \alpha_s - dz \sin \alpha_s \end{pmatrix}$$

Finally, we get the coordinates of O in a coordinate system where the sheet axis dips below the reference plane by the angle  $\theta$ . This corresponds to the actual geometric relation between the sheet and the observation point. We need to perform the rotation  $R(-\theta, x)$  around the local x axis, which lies along the radius of the reference circle. We find

$$O''' = \begin{pmatrix} -dx \sin \alpha_s - dz \cos \alpha_s \\ -(dx \cos \alpha_s - dz \sin \alpha_s) \sin \theta + dy \cos \theta \\ (dx \cos \alpha_s - dz \sin \alpha_s) \cos \theta + dy \sin \theta \end{pmatrix}$$

In this coordinate system the projection of the observation point on the z axis is the axial coordinate  $u$  needed by the sheet function. The two orthogonal coordinates of the observation point give the radial coordinate.

Now let us specialize to the case considered here, where  $dy = 0$ .

$$\begin{aligned} u &= O_z''' (dy = 0) = (dx \cos \alpha_s - dz \sin \alpha_s) \cos \theta \\ v_x &= -O_x''' = dx \sin \alpha_s + dz \cos \alpha_s \\ v_y &= O_y''' (dy = 0) = (-dx \cos \alpha_s + dz \sin \alpha_s) \sin \theta \\ v &= \sqrt{v_x^2 + v_y^2} \end{aligned}$$

It is not difficult to show that in the case when  $\theta = 0$  these expressions for  $u$  and  $v$  reduce to Eq. 1.

Now that we have  $u$  and  $v$ , we can evaluate  $BS(u,v)$  to find the field components  $B_U$  and  $B_V$  at the observation point. In Cartesian coordinates it has the components

$$B''' = \begin{pmatrix} B_{VX} \\ B_{VY} \\ B_U \end{pmatrix} = \begin{pmatrix} -B_V \frac{v_x}{v} \\ B_V \frac{v_y}{v} \\ B_U \end{pmatrix}$$

We next consider the field components in the coordinate system where the sheet axis is back in the reference plane. We rotate  $B'''$  by  $R(\theta, x)$  to find

$$B'' = \begin{pmatrix} B_{VX} \\ B_{VY} \cos \theta + B_U \sin \theta \\ -B_{VY} \sin \theta + B_U \cos \theta \end{pmatrix}$$

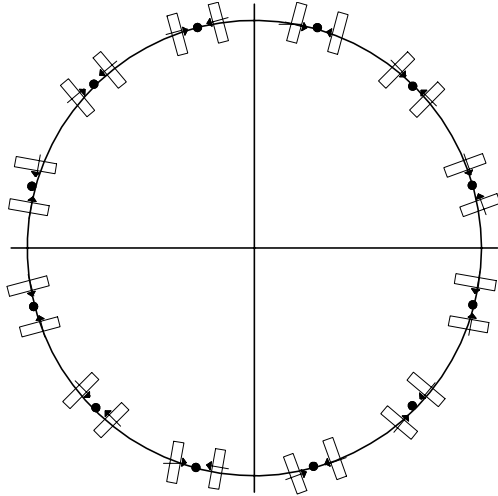
In the actual geometric configuration the observation point is separated from the center of the sheet by an angle  $\beta = \alpha_O - \alpha_S$ . Thus we need a final rotation  $R(-\beta, y)$ . The field components at O are

$$B = \begin{pmatrix} B_{VX} \cos \beta - (-B_{VY} \sin \theta + B_U \cos \theta) \sin \beta \\ B_{VY} \cos \theta + B_U \sin \theta \\ B_{VX} \sin \beta + (-B_{VY} \sin \theta + B_U \cos \theta) \cos \beta \end{pmatrix}$$

Note that this coordinate system has its x axis pointing radially outwards at the observation point. We must reverse the sign of  $B_X$  given above if we want the result in the accelerator system. It is not difficult to show that in the case when  $\theta = 0$  these expressions for  $B_X$  and  $B_S$  reduce to Eq. 2.

#### 4. Field along the circle in the RFOFO model

Now consider the distribution of sheets for the RFOFO cooling ring, shown in Fig. 5.



**Figure 5.** Layout of 24 current sheets for RFOFO model.

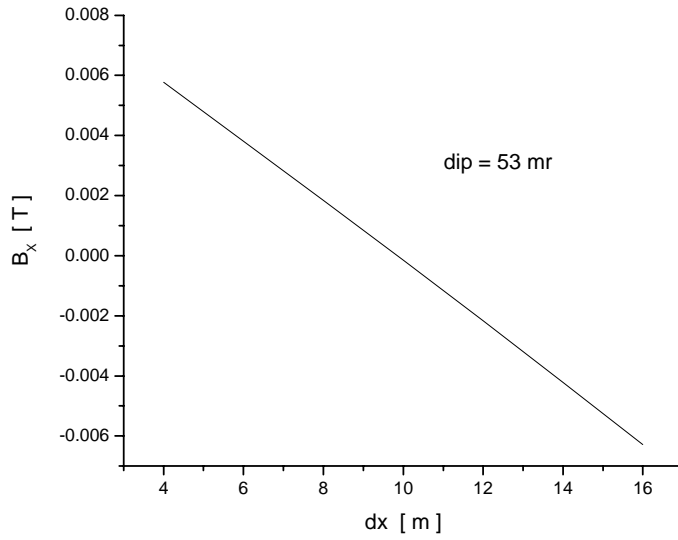
There are 12 identical cells, each consisting of two solenoids. We model each solenoid with a single current sheet, so there are 24 sheets in all to describe the whole ring. The properties of the sheets in each cell are given in Table 2.

*Table 2: Properties of sheets in each cell of RFOFO ring.*

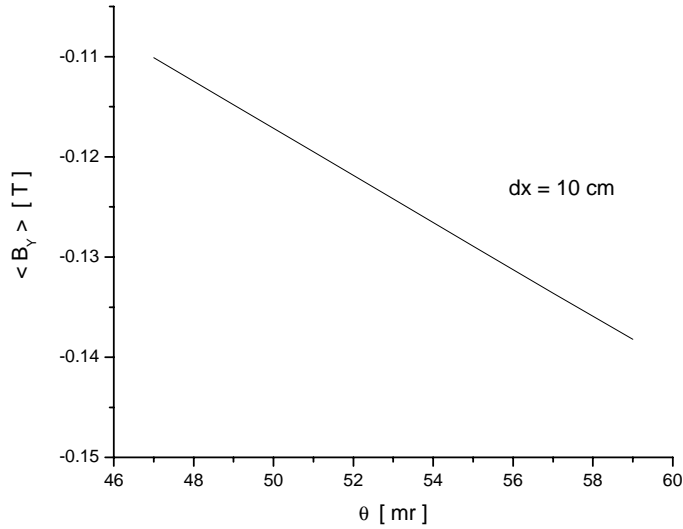
sheet	SS [ m ]	L [ m ]	a [ m ]	J [ A/mm <sup>2</sup> ]	$\theta$ [mr]	$\delta x$ [m]
1	0.30	0.50	0.825	95.27	53	0.10
2	1.95	0.50	0.825	-95.27	-53	0.10

SS is the axial position relative to the start of a cell of the sheet and  $\delta x$  is a radial displacement (positive outwards) of the sheet center. The radial displacement minimizes

the integral  $B_x$  along the circle for one cell, as shown in Fig. 6. The dip angle was chosen to give an average vertical field of -0.125 T over one cell, as shown in Fig. 7.

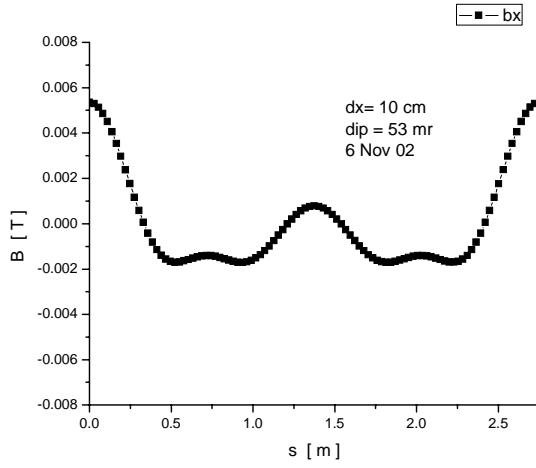


**Figure 6.** Integrated radial component of magnetic field versus radial displacement of sheet center.

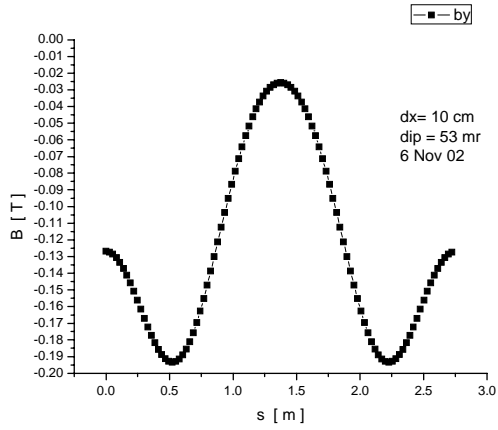


**Figure 7.** Vertical component of magnetic field versus dip angle of solenoid axis.

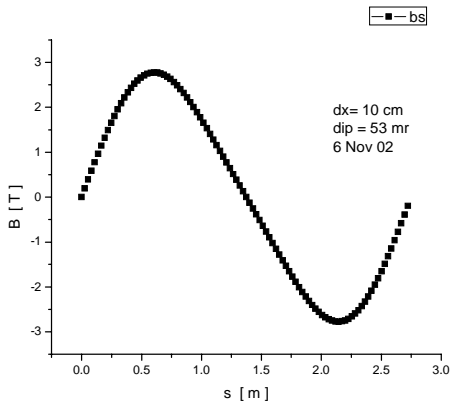
The calculated field components along the circle for one cell of the lattice are shown in Figs. 8-10.



**Figure 8.** Radial component of magnetic field in accelerator coordinates.



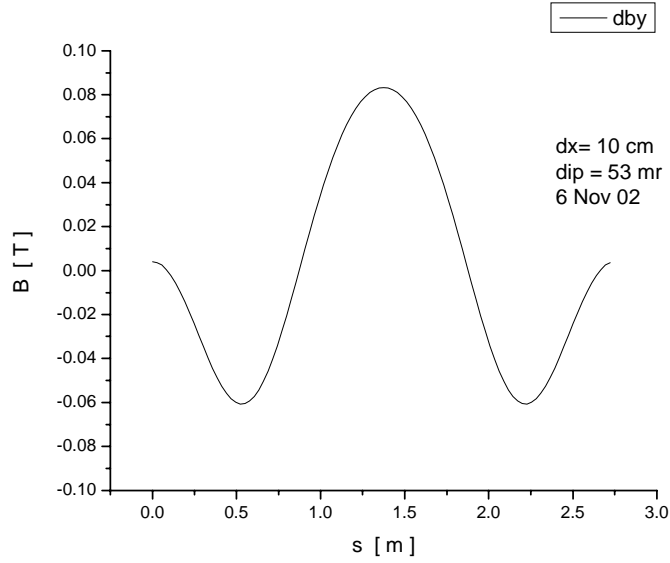
**Figure 9.** Vertical component of magnetic field in accelerator coordinates.



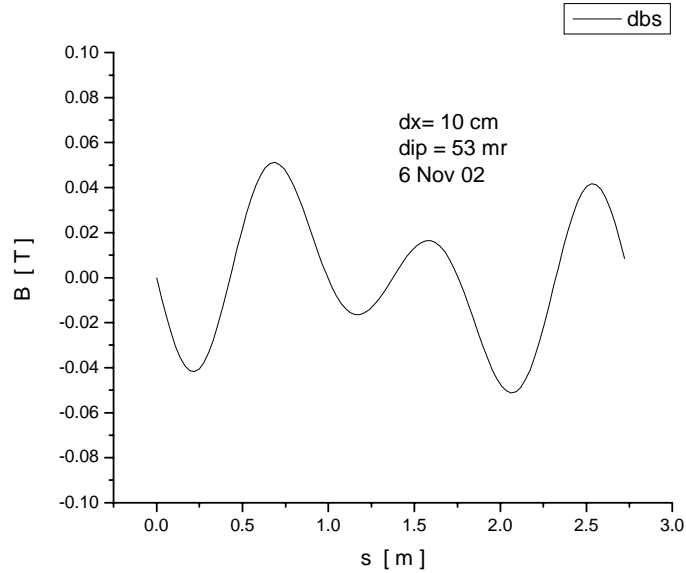
**Figure 10.** Axial component of magnetic field in accelerator coordinates.

Independent calculations of the field components made using the Biot-Savart equation gave essentially identical results [6].

Besides the presence of the  $B_x$  component, the deviations from the field components assumed in the *nglr* model are shown in Figs. 11 and 12.



**Figure 11.** Deviation of vertical field from that used in the *nglr* model.



**Figure 12.** Deviation of axial field from that used in the *nglr* model.

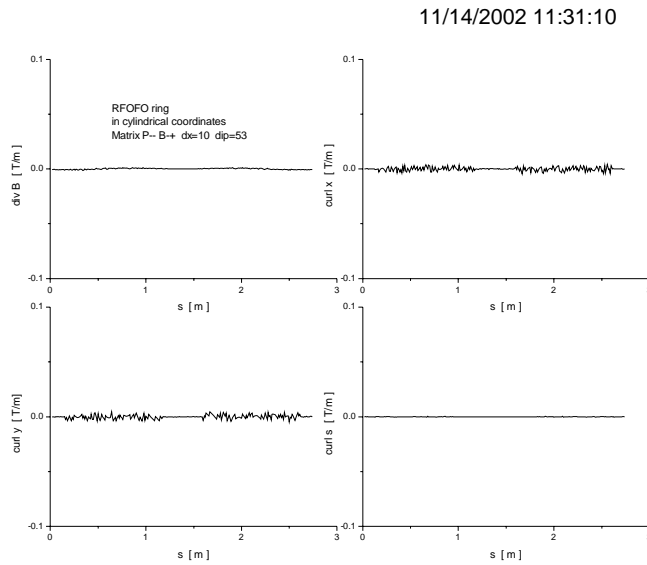
## 5. Check of Maxwell's equations

The ultimate test for the calculated field components is that they must satisfy the Maxwell equations

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{B} = 0$$

for points along the circle. We have computed these quantities numerically in Fig. 13. We use the symmetric form of the derivative, a cylindrical coordinate system and spacing 1 cm along  $s$  and 1 mm along  $x$  and  $y$ .



**Figure 13.** Divergence and curl components of the magnetic field for one period of the circular path around the RFOFO ring.

The divergence and curl were also checked in Cartesian coordinates and give similar agreement [6].

## 6. Conclusions

We have determined the magnetic field components on the SA for an RFOFO ring containing tipped solenoids. The field configuration assumes that no iron is present to shield points on the axis from solenoids at other points in the ring. The field components satisfy the Maxwell divergence and curl relations.

## Acknowledgements

We would like to thank Steve Bracker, Lucien Cremaldi and Bob Palmer for useful discussions.

## References

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